STABILITY OF A SOLID IN A VORTEX FLOW OF IDEAL LIQUID

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The problem of movement of a solid in an ideal liquid is a classical section of hydrodynamics [1, 2]. The stability of steady-state body movements in potential flows has been studied previously in [1-5]. In the present work the two-dimensional problem is considered for stability of a solid in a steady-state vortex flow of ideal incompressible liquid. A preservation functional is constructed which has a critical point in solving the steady-state problem of flow round a body. Adequate stability conditions are obtained by the Arnold method [6] for linear approximation. The general result is used for studying flow stability with circular flow lines in the case when an inner cylinder may move under the action of the forces of pressure from the direction of the liquid.

1. Statement of the Problem. The two-dimensional problem of movement of a solid in an ideal incompressible uniform liquid is considered. Movement of the body occurs in (m + 1)-connected region τ totally filled with liquid. Boundary $\partial\tau$ of region τ consists of m boundaries $\partial\tau_i$ (i = 1, ..., m) of singly-connected regions τ_i and outer boundary $\partial\tau_0$. At instant of time t the body occupies the region $\tau_b(t)$ within region τ.

On a Cartesian coordinate system x, y the equation of liquid motion has the form

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -(1/\rho)\,\nabla p - \nabla\Phi, \quad \operatorname{div}\mathbf{u} = 0, \quad \omega_t + (\mathbf{u}\cdot\nabla)\,\omega = 0. \tag{1.1}$$

Here $\mathbf{u} \equiv (u, v)$, p, $\omega \equiv v_x - u_y$ are fields of velocity, pressure, and vorticity; ρ is liquid density; Φ is potential of external forces operating on the liquid.

Body movement is described by the equations

$$m\dot{V}_i \equiv m\ddot{R}_i = -\int\limits_{\partial\tau_b} pn_i dS - \frac{\partial\Phi_b(\mathbf{R},\varphi)}{\partial R_i},$$
$$I\dot{\Omega}_b \equiv I\ddot{\varphi} = -\int\limits_{\partial\tau_b} \mathbf{z}\cdot[(\mathbf{r}-\mathbf{R})\times\mathbf{n}]\,pdS - \frac{\partial\Phi_b(\mathbf{R},\varphi)}{\partial\varphi}, \tag{1.2}$$

where **z** is unit vector in the direction of axis z; m is body mass; I is body moment of inertia; **V** is body forward movement velocity; Ω_b is angular velocity of rotation of the body around axis z; **R** is radius vector of the body center of mass; φ is an angular variable prescribing body orientation; Φ_b is potential of external forces operating on the body.

At boundaries $\partial\tau_k$ (k = 0, ..., m) and $\partial\tau_b$ the normal condition of no flow is set:

$$\mathbf{u}\cdot\mathbf{n} = \{\mathbf{V} + \Omega_b[\mathbf{z}\times(\mathbf{r}-\mathbf{R})]\}\cdot\mathbf{n} \quad \text{at} \quad \partial\tau_b,$$
$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{at} \quad \partial\tau_k, \quad k = 0, \ldots, m. \tag{1.3}$$

Here **n** are external normals to $\partial\tau_b$ and $\partial\tau_k$; **r** is radius vector of a point on the body surface $\partial\tau_b$.

Integrals are retained in the solutions of problems (1.1)-(1.3)

$$E = \int\limits_{\tau-\tau_b} \left\{\rho\frac{u_i u_i}{2} + \rho\Phi\right\} d\tau + \frac{1}{2}m\dot{R}_i\dot{R}_i + \frac{1}{2}I\dot{\varphi}^2 + \Phi_b(\mathbf{R},\varphi); \tag{1.4}$$

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$$C = \int_{\tau - \tau_b} F(\omega)\, d\tau, \tag{1.5}$$

where F(ω) is an arbitrary function; summing is carried out by repeating vector indices. Integral (1.4) is the total energy of the body–liquid system, and (1.5) is a consequence of retaining vorticity in each liquid particle. In addition, in view of the Kelvin theorem there is also retention of velocity circulation with respect to closed curves $\partial\tau_k$, $\partial\tau_b$:

$$\Gamma_k = \int_{\partial\tau_k} \mathbf{u}\cdot\sigma dS \quad (k = 0, ..., m), \quad \Gamma_b = \int_{\partial\tau_b} \mathbf{u}\cdot\sigma\, dS \tag{1.6}$$

(σ is tangential vector to the curve for which integration is performed).

Then the problem is considered for the stability of accurate solution of problem (1.1)-(1.3) corresponding to a steady-state regime of flow round a body. On the coordinate system connected with the body (the origin coincides with the body center of mass) this solution has the form

$$R_i = \dot{R}_i = \dot{\varphi} = 0, \quad u_i = U_i(\mathbf{x}) \text{ in } \tau - \tau_b. \tag{1.7}$$

The velocity field **U(x)** is the solution of the flow problem:

$$(\mathbf{U}\cdot\nabla)\,\mathbf{U} = -\frac{1}{\rho}\nabla P + \nabla\Phi, \quad \operatorname{div}\mathbf{U} = 0 \text{ in } \tau - \tau_b,$$
$$\mathbf{U}\cdot\mathbf{n} = 0 \text{ at } \partial\tau_b \text{ and } \partial\tau_k \quad (k = 0, ..., m). \tag{1.8}$$

Forces acting on the body from the direction of the liquid and the moment of forces are balanced by an external force and moment:

$$\frac{\partial\Phi_b}{\partial R_i} = -\int_{\partial\tau_b} n_i P dS; \tag{1.9}$$

$$\frac{\partial\Phi_b}{\partial\varphi} = -\int_{\partial\tau_b} \mathbf{z}\cdot(\mathbf{r}\times\mathbf{n})\, P dS. \tag{1.10}$$

A preservation functional is composed from integrals (1.4)-(1.6)

$$I = \int_{\tau-\tau_b} \rho\left\{\frac{u_i u_i}{2} + \Phi + F(\omega)\right\} d\tau + \frac{m}{2}\mathbf{V}^2 + \frac{I}{2}\Omega_b^2 + \Phi_b + \sum_{k=0}^{m} A_k\Gamma_k + B\Gamma_b \tag{1.11}$$

(B, A_k (k = 0, ..., m) are arbitrary constants).

It will be shown below that with appropriate selection of function F(ω) and constants B, A_k solution (1.7) is a steady-state point of functional I.

2. Extremum Conditions. For the first variation of functional (1.11) taken in solving (1.7) the following representation is correct:

$$\delta I = \int_{\tau-\tau_b} \rho\,\{\mathbf{U} + \operatorname{rot}[F'(\Omega)\mathbf{z}]\}\cdot\delta\mathbf{u}\, d\tau + m\mathbf{V}\cdot\delta\mathbf{V} + I\Omega_b\delta\Omega_b +$$
$$+ \left\{\frac{\partial\Phi_b}{\partial R_i} - \int_{\partial\tau_b} \rho n_i\left[\frac{U_i U_i}{2} + \Phi\right] dS\right\}\delta R_i + \left\{\frac{\partial\Phi_b}{\partial\varphi} - \int_{\partial\tau_b} \rho\mathbf{z}\cdot(\mathbf{r}\times\mathbf{n})\left[\frac{U_i U_i}{2} + \Phi\right] dS\right\}\delta\varphi +$$
$$+ [A_0 + \rho F'(\Omega_0)]\int_{\partial\tau_0} (\delta\mathbf{u}\cdot\sigma)\, dS + \sum_{k=1}^{m}[A_k - \rho F'(\Omega_k)]\int_{\partial\tau_k} (\delta\mathbf{u}\cdot\sigma)\, dS +$$
$$+ [B - \rho F'(\tilde{\Omega})]\int_{\partial\tau_b} (\delta\mathbf{u}\cdot\sigma)\, dS. \tag{2.1}$$

Here $\delta\mathbf{r} = \delta\mathbf{R} + \delta\varphi\ (\mathbf{z} \times \mathbf{r})$ is infinitely small displacement of point $\mathbf{r}$ at the body surface with variation; $\delta\mathbf{R}$ is body displacement as a whole; $\delta\varphi$ is body rotation around axis z; Ω is vorticity of the main flow; $\Omega_k \equiv \Omega$ on $\partial\tau_k$; $\tilde{\Omega} \equiv \Omega$ on $\partial\tau_b$.

It can be seen from (2.1) that $\delta I = 0$ if the following conditions are fulfilled:

$$A_k = \rho F'(\Omega_k), \quad k = 1, \ldots, m; \quad A_0 = -\rho F'(\Omega_0); \quad B = \rho F'(\tilde{\Omega}); \tag{2.2a}$$

$$\dot{R}_i = V_i = 0, \quad \dot{\varphi} = \Omega_b = 0; \tag{2.2b}$$

$$\mathrm{U} = -\operatorname{rot}[F'(\Omega)\,\mathrm{z}]; \tag{2.2c}$$

$$\frac{\partial \Phi_b}{\partial R_i} = \int\limits_{\partial\tau_b} \rho n_i \left[\frac{U_i U_i}{2} + \Phi\right] dS; \tag{2.2d}$$

$$\frac{\partial \Phi_b}{\partial \varphi} = \int\limits_{\partial\tau_b} \rho \mathrm{z} \cdot (\mathrm{r} \times \mathrm{n}) \left[\frac{U_i U_i}{2} + \Phi\right] dS. \tag{2.2e}$$

We select constants B, A_k (k = 0, ..., m) so that condition (2.2a) is fulfilled. Equality (2.2b) is always fulfilled in steady-state solution (1.7). If the function of point Ψ of the main flow is determined

$$\mathrm{U}(\mathrm{x}) = -\operatorname{rot}(\Psi \mathrm{z}), \tag{2.3}$$

then (2.2c) means that in solution (1.7)

$$\Psi = F'(\Omega). \tag{2.4}$$

By using (2.3) Eqs. (1.8) may be rewritten in the form

$$-\Omega \nabla \Psi = -\nabla(P/\rho + \Phi + U_i U_i/2).$$

Taking account of (2.4) it follows that

$$\Phi + \frac{U_i U_i}{2} = -\frac{P}{\rho} + H(\Omega) + \text{const}, \quad \frac{dH}{d\Omega} = \Omega \frac{d^2 F}{d\Omega^2}. \tag{2.5}$$

By substituting (2.5) in Eqs. (2.2d) and (2.2e) it is possible to see that they coincide with conditions for body equilibrium (1.9) and (1.10).

Thus, it is shown that in the set of functions $\mathbf{u}(\mathbf{x}, t)$, $\mathbf{R}(t)$, $\mathbf{V}(t)$, $\varphi(t)$, $\Omega_b(t)$ which satisfy the nonflow condition (1.3) solutions (1.7) of problem (1.8) are stationary points of functional (1.11). (Here it is assumed that the required properties for the smoothness of function $\mathbf{u}(\mathbf{x}, t)$ are fulfilled.) Thereby a generalized Arnold result [6] is given for the case of presence of a solid in the liquid.

In order to explain the nature of the critical point of functional I we work out its second variation at this point:

$$\begin{aligned} \delta^2 I = \int\limits_{\tau-\tau_b} \rho \left[(\delta \mathrm{u})^2 + \frac{d\Psi}{d\Omega}(\delta\omega)^2\right] d\tau + m(\delta\dot{\mathrm{R}})^2 + I(\delta\dot{\varphi})^2 - \\ - \int\limits_{\partial\tau_b} \rho(\delta \mathrm{r} \cdot \mathrm{n})\,[2\mathrm{U} \cdot \delta \mathrm{u} + (\delta \mathrm{r} \cdot \nabla) G]\, dS - \\ - \int\limits_{\partial\tau_b} \rho \delta\varphi\,(\delta \mathrm{R} \cdot \sigma)\, G\, dS + \frac{\partial^2 \Phi_b}{\partial q_i \partial q_j} \delta q_i \delta q_j. \end{aligned} \tag{2.6}$$

Here δu_i, $\delta\omega$, δR_i, $\delta\varphi$ are variations of the corresponding values; $\delta\mathbf{r} \equiv \delta\mathbf{R} + \delta\varphi\ [\mathbf{z} \times \mathbf{r}]$; $\delta q_i \equiv (\delta R_1, \delta R_2, \delta\varphi)$; $G \equiv U_i U_i/2 + \Phi$.

3. Linearized Problem Integrals. Equations of motion linearized in solution (1.7) have the form

$$\left.\begin{aligned} &D\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{U} = -\frac{1}{\rho}\nabla p \\ &D\omega + (\mathbf{u}\cdot\nabla)\,\Omega = 0 \\ &\operatorname{div}\mathbf{u} = 0,\quad D \equiv \frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla \end{aligned}\right\} \text{ in } \quad \tau - \tau_b; \tag{3.1}$$

$$m\ddot{R}_i = -\int_{\partial\tau_b} \rho\,[(p + \mathbf{r}\cdot\nabla P)\,n_i + \varphi P\sigma_i]\,dS + \frac{\partial\Phi_b}{\partial R_{0_i}} R_i; \tag{3.2}$$

$$I\ddot{\varphi} = -\int_{\partial\tau_b} \rho\,[\mathbf{z}\cdot(\mathbf{r}_0\times\mathbf{n})]\,(p + \mathbf{r}\cdot\nabla P)\,dS + \frac{\partial\Phi_b}{\partial\varphi_{0_i}}\varphi, \tag{3.3}$$

where u_i, ω, ρ, R_i, φ are infinitely small disturbances of the corresponding values; $\mathbf{r} \equiv \mathbf{R} + \varphi\,[\mathbf{z}\times\mathbf{r}_0]$; in (3.2) and (3.3) integration is performed with respect to the undisturbed surface of the body $\partial\tau_b$.

In linearizing problems with an unknown moving boundary on Euler coordinates difficulties arise connected with 'removal' of boundary conditions from the disturbed surface to the undisturbed surface. Therefore, here it is convenient to use the linearization method given in [7, 8]. Disturbed flow $\mathbf{X}(\mathbf{a}, t)$ ($\mathbf{a}$ is Lagrangian coordinate) is broken down into two parts

$$\mathbf{X}(\mathbf{a}, t) = \mathbf{x}(\mathbf{a}, t) + \boldsymbol{\xi}(\mathbf{x}(\mathbf{a}, t), t) \tag{3.4}$$

[$\mathbf{x}(\mathbf{a}, t)$ is undisturbed flow]. In view of the continuity condition for flow the linearized boundary condition at $\partial\tau_b$ is as follows: $(\mathbf{X}(\mathbf{a}, t) - \mathbf{x}(\mathbf{a}, t))\cdot\mathbf{n} = \mathbf{r}\cdot\mathbf{n}$. Taking account of (3.4) it takes the form

$$\boldsymbol{\xi}(\mathbf{x}, t)\cdot\mathbf{n} = \mathbf{r}\cdot\mathbf{n}. \tag{3.5}$$

The connection of Euler disturbances of velocity $\mathbf{u}(\mathbf{x}, t)$ with Lagrangian displacements of liquid particles $\xi(\mathbf{x}, t)$ is given by the equation [7, 8]

$$D\boldsymbol{\xi} = \mathbf{u} + (\boldsymbol{\xi}\cdot\nabla)\,\mathbf{U}. \tag{3.6}$$

The boundary condition for the velocity field follows from (3.5) and (3.6)

$$\mathbf{u}\cdot\mathbf{n} = D(\mathbf{r}\cdot\mathbf{n}) + (\mathbf{r}\cdot\mathbf{n})\frac{1}{Q}DQ \quad \text{at} \quad \partial\tau_b \quad (Q^2 \equiv U_iU_i). \tag{3.7}$$

At stationary boundaries $\partial\tau_k$ ($k = 0, \ldots, m$) linearized boundary conditions have the normal form

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{at} \quad \partial\tau_k. \tag{3.8}$$

It is well known [6, 7] that there is preservation of the second variation of (2.6) in view of linearized problems (3.1)-(3.3), (3.7), (3.8), which may be confirmed by direct calculations. Variations $\delta\mathbf{u}$, $\delta\omega$, $\delta\mathbf{R}$, $\delta\varphi$ imply infinitely small disturbances of $\mathbf{u}$, ω, $\mathbf{R}$, φ, which satisfy Eqs. (3.1)-(3.3). In accordance with this in (2.6) we make the following redesignations: $\delta\mathbf{u} \to \mathbf{u}$, $\delta\omega \to \omega$, $\mathbf{r} \to \mathbf{r}_0$, $\delta\mathbf{r} \to \mathbf{r}$, $\delta q_i \to q_i$. Equation (2.6) is written as

$$\begin{aligned} E_1 \equiv \frac{1}{2}\delta^2 I = \int_{\tau-\tau_b} \rho\left[\frac{u_iu_i}{2} + \frac{d\Phi}{d\Omega}\frac{\omega^2}{2}\right]d\tau + \frac{m}{2}\dot{\mathbf{R}}^2 + \frac{I}{2}\dot{\varphi}^2 - \\ - \int_{\partial\tau_b} \rho\,(\mathbf{r}\cdot\mathbf{n})\left[\mathbf{U}\cdot\mathbf{u} + \frac{1}{2}(\mathbf{r}\cdot\nabla)\,G\right]dS - \int_{\partial\tau_b} \frac{\rho}{2}\varphi\,(\mathbf{R}\cdot\sigma)\,G\,dS + \frac{1}{2}\frac{\partial^2\Phi_b}{\partial q_{0_i}\partial q_{0_k}}q_iq_k. \end{aligned} \tag{3.9}$$

When there is positive definiteness of E_1 as a quadratic form of $\mathbf{u}$, $\mathbf{R}$, $\dot{\mathbf{R}}$, φ, $\dot{\varphi}$ from the equality E_1 = const stability of solution (1.7) emerges for linear approximation. In fact, if deviations of disturbed flow from undisturbed integral E_1 are measured, then there is stability in the Lyapunov determination: for any number $\varepsilon > 0$ another number $\delta > 0$ is found so that only $E_1(0) < \delta$, and then for all $t > 0$ the condition $E_1(t) < \varepsilon$ is fulfilled. Here it is sufficient to take $\delta = \varepsilon$.

Energy integral (3.9) is only determined when in the whole of the flow region $\tau - \tau_b$ there is fulfillment of the condition $\Omega' \equiv (d\Omega/d\Psi) \neq 0$. For the important class of flows with constant vorticity $\Omega' \equiv 0$ Eq. (3.9) does not make sense. In this case of the linear problem integrals a reduction factor may be obtained for the class of disturbances. For this equations in vortex disturbance (3.1) using (3.6) are reduced to the form

$$D\,(\omega + \xi \cdot \nabla\Omega) = 0. \tag{3.10}$$

It follows from (3.10) that if in the initial instant of time it is chosen that

$$\omega = -\xi \cdot \nabla\Omega = -\,\Omega'\,(\xi \cdot \nabla\Psi), \tag{3.11}$$

then equality (3.11) will be fulfilled with all t. Equation (3.11) means limitation of the class of disturbances to the so-called 'equal vortex' class [9]. For these disturbances the magnitude of the vortex is constant in each liquid particle and the field for the vortex only changes as a result of movement of these particles. Integral E_1 (3.9) for a narrower class of disturbances remains correct only if the integral for region $\tau - \tau_b$ in accordance with (3.11) which is in expression (3.9) takes the form

$$\int\limits_{\tau-\tau_b} \rho \left[\frac{u_i u_i}{2} + \frac{1}{2}\,\Omega'\,(\xi \cdot \nabla\Psi)^2\right] d\tau. \tag{3.12}$$

With $\Omega' = 0$ it follows from (3.11) that $\omega = 0$ and the energy integral E_1 taking account of (3.12) is reduced to the form

$$\begin{aligned} E_1 = \int\limits_{\tau-\tau_b} \rho \frac{u_i u_i}{2}\, d\tau + \frac{m}{2}\dot{R}^2 + \frac{I}{2}\dot{\varphi}^2 + \frac{1}{2}\frac{\partial^2 \Phi_b}{\partial q_{0_i} \partial q_{0_k}}\, q_i q_k \,- \\ - \int\limits_{\partial\tau_b} \rho\,(r \cdot n) \left[U \cdot u + \frac{1}{2}(r \cdot \nabla)\, G\right] dS - \int\limits_{\partial\tau_b} \frac{\rho}{2}\,\varphi\,(R \cdot \sigma)\, G\, dS. \end{aligned} \tag{3.13}$$

In accordance with this integral E_1 (3.13) is preserved in view of the linear problem if the field for disturbance velocity is considered potential.

4. Sufficient Conditions for Stability. As has been shown above, sufficient conditions for stability of solution (1.7) coincide with conditions for energy integral (3.9) as a quadratic form of u_i, $\dot{R}_i$, R_i, $\dot{\varphi}$, φ having fixed sign. In order to reduce E_1 (3.9) to a form convenient for studying its property of having a fixed sign, in region $\tau - \tau_b$ a subsidiary vector field $\alpha(x)$ is introduced. It is assumed that the properties of, function smoothness for $\alpha(x)$ required subsequently are fulfilled. It is possible to show that for any field $\alpha(x)$ such that $\alpha = 0$ at $\partial\tau$, $\alpha = \alpha\sigma$ at $\partial\tau_b$, the following equality is correct:

$$\int\limits_{\tau-\tau_b} \{[2e_{jil} z_l \partial_j \alpha_k - z \cdot \text{rot}\,\alpha\,\delta_{ik}]\, u_i u_k + 2\omega u_k \alpha_k\}\, d\tau + \int\limits_{\partial\tau_b} \alpha\,\{(u \cdot \sigma)^2 - (u \cdot n)^2\}\, dS = 0. \tag{4.1}$$

By combining expressions (3.9) and (4.1) and separating perfect squares in integrals for region $\tau - \tau_b$ and for boundary $\partial\tau_b$ the expression (3.1) may be converted to the form

$$\begin{aligned} E_1 = K^* + \frac{1}{2}\int\limits_{\partial\tau_b} \rho\alpha \left(u \cdot \sigma - \frac{U \cdot \sigma}{\alpha}\, r \cdot n\right)^2 dS + W, \\ 2W = (m\delta_{ik} - F_{ik})\,\dot{R}_i \dot{R}_k - 2Q_{ik}\dot{R}_i R_k + (a_{ik} + A_{ik})\, R_i R_k + (I - D)\,\dot{\varphi}^2 + \\ + (c + C)\,\varphi^2 - 2H\varphi\dot{\varphi} - 2L_i \dot{R}_i \dot{\varphi} + 2N_i \dot{R}_i \varphi + 2M_i R_i \dot{\varphi} + 2\,(b_i - B_i)\, R_i \varphi. \end{aligned} \tag{4.2}$$

Here

$$2K^* = \int\limits_{\tau-\tau_b} \rho \left\{\lambda_{ik} u_i u_k + \frac{d\Psi}{d\Omega}\left(\omega + \frac{d\Omega}{d\Psi}\, u \cdot \alpha\right)^2\right\} d\tau;$$

$$\lambda_{ik} \equiv (1 - z \cdot \text{rot}\,\alpha)\,\delta_{ik} + 2e_{jil} z_l \partial_j \alpha_k - \frac{d\Omega}{d\Psi}\,\alpha_i \alpha_k;$$

$$a_{ik} \equiv \frac{\partial^2 \Phi_b}{\partial R_i\, \partial R_k}; \quad b_i \equiv \frac{\partial^2 \Phi_b}{\partial R_i\, \partial \varphi}; \quad c \equiv \frac{\partial^2 \Phi_b}{\partial \varphi^2};$$

$$F_{ik} \equiv \int_{\partial\tau_b} \rho\alpha n_i n_k\, dS; \quad Q_{ik} \equiv \int_{\partial\tau_b} \rho\alpha n_i \Pi_k\, dS; \quad \Pi_k \equiv (\sigma\cdot\nabla)\, \partial_k \Psi;$$

$$A_{ik} \equiv -\int_{\partial\tau_b} \rho \left\{ n_i \partial_k G + \frac{U_l U_l}{\alpha} n_i n_k + \alpha \Pi_i \Pi_k \right\} dS;$$

$$B_i \equiv \int_{\partial\tau_b} \rho \left\{ -\alpha \Pi_i \Lambda - \frac{U_k U_k}{\alpha} (r_0\cdot\sigma)\, n_i + \frac{\alpha_i G}{2} + \frac{n_i}{2} (z\cdot [r_0 \times \nabla G]) - \frac{r_0\cdot\sigma}{2} \partial_i G \right\} dS;$$

$$C \equiv \int_{\partial\tau_b} \rho \left\{ r_0\cdot\sigma\, (z\cdot [r_0 \times \nabla G]) - \alpha \Lambda^2 - \frac{U_k U_k}{\alpha} (r_0\cdot\sigma)^2 \right\} dS; \quad \Lambda \equiv (\sigma\cdot\nabla)\,(r_0\cdot U);$$

$$D \equiv \int_{\partial\tau_b} \rho\alpha\, (r_0\cdot\sigma)^2\, dS; \quad H \equiv \int_{\partial\tau_b} \rho\alpha (r_0\cdot\sigma)\, \Lambda\, dS; \quad L_i \equiv \int_{\partial\tau_b} \rho\alpha\, (r_0\cdot\sigma)\, n_i\, dS;$$

$$N_i \equiv \int_{\partial\tau_b} \rho\alpha \Lambda n_i\, dS; \quad M_i \equiv \int_{\partial\tau_b} \rho\alpha\, (r_0\cdot\sigma)\, \Pi_i\, dS.$$

Energy integral (4.2) is determined positively if simultaneously the following conditions are fulfilled:

$$\frac{d\Psi}{d\Omega} > 0 \quad \text{in} \quad \tau - \tau_b; \tag{4.3}$$

$$\lambda_{ik} u_i u_k > 0 \quad \text{in} \quad \tau - \tau_b; \tag{4.4}$$

$$\alpha > 0 \quad \text{at} \quad \partial\tau_b; \tag{4.5}$$

$$W > 0. \tag{4.6}$$

Condition (4.3) coincides with the sufficient condition for Arnold stability for plane flows of an ideal incompressible liquid in a fixed region [6]. Let it be fulfilled. By using arbitrary α we select it so that conditions (4.4) and (4.5) are fulfilled. This is always possible by choosing a quite small field for α. Now the condition for E_1 having a fixed sign coincides with conditions for a quadratic form of W from $\dot{R}_i$, R_i, $\dot{\varphi}$, φ having fixed sign. It follows from the form of W (4.2) that with quite small α (such that $\max\{F_{11}, F_{22}\} < m$, $D < I$) it is possible to find such a_{ik}, b_i, c for which the quadratic form of W will be determined positively. Thus, the following is correct: if for basic solution (1.7) the Arnold stability condition is fulfilled

$$\frac{d\Psi}{d\Omega} > 0 \quad \text{in} \quad \tau - \tau_b,$$

then there will always exist such a potential Φ_b of external forces which operate on the body that solution (1.7) will be stable in the sense of preserving energy integral E_1 (4.2). This means that external forces which operate on the body may stabilize its movement in a vortex flow satisfying the Arnold stability condition.

It is possible to formulate clearly energy conditions which should satisfy the potential of external forces Φ_b sufficient for positive determination of the energy integral. In order to avoid cumbersome computations we dwell on the particular case when the body is a round cylinder of radius r_1. Then generalized coordinate φ is cyclic and those disturbances which are turnings of the body cannot be considered. Energy integral E_1 (3.9) takes the form

$$E_1 = \int_{\tau-\tau_b} \rho \left[\frac{u_i u_i}{2} + \frac{d\Psi}{d\Omega} \frac{\omega^2}{2} \right] d\tau - \int_{\partial\tau_b} \rho\, (R\cdot n) \left[U\cdot u + \frac{1}{2} R\cdot\nabla G \right] dS + \tag{4.7}$$

$$+ \frac{m}{2} \dot{R}_i \dot{R}_i + \frac{1}{2} a_{ik} R_i R_k.$$

We introduce a polar coordinate system (r, ϑ). Let (U, V) and (u, v) be the corresponding components of the velocity field for the basic flow and disturbances. We assume that $\alpha = r_1\tilde{\alpha}$ = const at $\partial\tau_b$. Then similar to (4.2) the expression for the energy integral may be written in the form

$$E_1 = K^* + \frac{1}{2}\int_0^{2\pi} \rho r_1^2\tilde{\alpha}\left(v - \frac{V}{r_1\tilde{\alpha}}\,\mathbf{R}\cdot\mathbf{n}\right)^2 d\vartheta + W,$$
$$2W = (m\delta_{ik} - F_{ik})\,\dot{R}_i\dot{R}_k - 2Q_{ik}\dot{R}_iR_k + (a_{ik} + A_{ik})\,R_iR_k. \qquad (4.8)$$

Since for a round cylinder $\mathbf{n}$ = (cos ϑ, sin ϑ), $\boldsymbol{\sigma}$ = (−sin ϑ, cos ϑ), then it is possible to obtain

$$F_{ik} = \tilde{\alpha}\mu\delta_{ik}, \quad \mu \equiv \rho\pi r_1^2, \quad Q_{ik} = \tilde{\alpha}\mu\tilde{Q}_{ik}, \quad \tilde{Q}_{ik} \equiv \frac{1}{\pi r_1}\int_0^{2\pi} Vn_i\sigma_k\,d\vartheta,$$
$$A_{ik} = \mu\left(A_{ik}^0 - \tilde{\alpha}A_{ik}^+ - \frac{1}{\tilde{\alpha}}A_{ik}^-\right), \quad A_{ik}^0 \equiv -\frac{1}{\pi r_1}\int_0^{2\pi} n_i\partial_kG\,d\vartheta,$$
$$A_{ik}^+ \equiv \frac{1}{\pi r_1^2}\int_0^{2\pi}\partial_\vartheta(n_iV)\,\partial_\vartheta(n_kV)\,d\vartheta, \quad A_{ik}^- \equiv \frac{1}{\pi r_1^2}\int_0^{2\pi} V^2n_in_k\,d\vartheta.$$

Consequently,

$$2W = (m - \tilde{\alpha}\mu)\,\dot{R}^2 - 2\tilde{\alpha}\mu\tilde{Q}_{ik}\dot{R}_iR_k + \mu\left(\frac{1}{\mu}a_{ik} + A_{ik}^0 - \tilde{\alpha}A_{ik}^+ - \frac{1}{\tilde{\alpha}}A_{ik}^-\right)R_iR_k. \qquad (4.9)$$

By separating the perfect square in (4.9) we have

$$2W = (m - \tilde{\alpha}\mu)\left(\dot{R}_i - \frac{\tilde{\alpha}\mu}{m - \tilde{\alpha}\mu}\tilde{Q}_{ik}R_k\right)^2 + 2\mu w_{ik}R_iR_k,$$
$$2w_{ik} \equiv \frac{1}{\mu}a_{ik} + A_{ik}^0 - \tilde{\alpha}A_{ik}^+ - \frac{1}{\tilde{\alpha}}A_{ik}^- - \frac{\tilde{\alpha}^2\mu}{m - \tilde{\alpha}\mu}\tilde{Q}_{li}\tilde{Q}_{lk}. \qquad (4.10)$$

It follows from (4.10) that the quadratic form of W is determined positively if the following conditions are fulfilled

$$\tilde{\alpha} \leqslant m/\mu; \qquad (4.11)$$

$$w_{ik}R_iR_k \geqslant 0. \qquad (4.12)$$

Thus, by selecting $\tilde{\alpha}$ so that (4.11) is fulfilled it is then possible for any solution in the form of (1.7) to check the correctness of inequality (4.12). If it is not fulfilled then there is stability in the sense of preserving integral (4.8).

5. Stability of Flow between Cylinders. We consider flow with circular current lines between cylinders. The inner cylinder is assumed to be unsecured. Its movement is described by equations of motion for a solid. Let r_1 and r_2 be cylinder inner and outer radii. The main solution (1.7) is thus:

$$\dot{\mathbf{R}}_0 = \mathbf{R}_0 = 0, \quad V = V(r), \quad \Omega = V'(r) + V(r)/r. \qquad (5.1)$$

Energy integral (4.7) may be written in the form

$$E_1 = \int_{\tau-\tau_b}\rho\left[\frac{u^2 + v^2}{2} + \frac{V}{\Omega'}\frac{\omega^2}{2}\right]d\tau + \frac{m}{2}\dot{\mathbf{R}}^2 - \int_0^{2\pi}\rho\,(\mathbf{R}\cdot\mathbf{n})\left[Vv + \frac{1}{2}(\mathbf{R}\cdot\mathbf{n})\,VV'\right]r_1\,d\vartheta. \qquad (5.2)$$

It is assumed that external forces are absent: $\Phi = \Phi_b = 0$.

Since the main solution (5.1) is invariant with respect to rotations around axis z, then in view of linearized equations of motion (3.1)-(3.3) and boundary conditions (3.7) and (3.8) integral M is also preserved, which is a second variation of the pulse for the body−liquid system:

$$M = \int\limits_{\tau-\tau_b} \rho \frac{r}{\Omega'} \frac{\omega^2}{2} d\tau - \int\limits_0^{2\pi} \rho (\mathbf{R}\cdot\mathbf{n}) \left[v + \frac{1}{2} (\mathbf{R}\cdot\mathbf{n}) \Omega \right] r_1^2 \, d\vartheta + mz\cdot(\mathbf{R} \times \dot{\mathbf{R}}) \tag{5.3}$$

(preservation of M may be demonstrated by direct calculations). From E_1 and M a preservation functional $\tilde{E} = E_1 + \lambda M$ (λ is an arbitrary constant) is composed. If we select $\lambda = -V(r_1)/r_1$, then $\tilde{E}$ takes the form

$$E = \int\limits_{\tau-\tau_b} \rho \left[\frac{u^2 + v^2}{2} + g(r) \frac{\omega^2}{2} \right] d\tau + \rho\pi r_1^2 \lambda^2 \left[1 - \frac{m}{\rho\pi r_1^2} \right] \frac{\mathbf{R}^2}{2} + \\ + \frac{m}{2} [\dot{R}_1 - \lambda R_2]^2 + \frac{m}{2} [\dot{R}_2 + \lambda R_1]^2, \tag{5.4}$$

where $g(r) \equiv (V(r) + \lambda r)/\Omega'(r)$; R_1, R_2 are components of vector **R** on Cartesian coordinate system x, y.

Integral (5.4) is determined positively and consequently there is stability if the following conditions are fulfilled

$$g(r) = \frac{V(r) - rV(r_1)/r_1}{\Omega'(r)} \geqslant 0, \quad r_1 \leqslant r \leqslant r_2; \tag{5.5}$$

$$m \leqslant \rho\pi r_1^2. \tag{5.6}$$

For flow with a constant vortex ($\Omega(r)$ = const), which corresponds to the velocity profile

$$V(r) = Ar + B/r, \tag{5.7}$$

a sufficient condition for stability in the class of potential disturbances is reduced to inequality (5.6).

Another sufficient condition for stability for flow (5.7) is obtained directly from conditions (4.11) and (4.12). It is possible to show that for (5.7) the sufficient condition of stability [in the sense of preserving integral (4.8)] consists of simultaneous fulfillment of the following inequalities:

$$-1 \leqslant r_1 \tilde{\alpha} r \left(\frac{1}{r} \chi' \right)' \leqslant 1; \tag{5.8}$$

$$r_1 V(r_1) V'(r_1) \leqslant -\gamma(\tilde{\alpha}) (V(r_1))^2, \quad \gamma(\tilde{\alpha}) \equiv \tilde{\alpha} + \frac{1}{\tilde{\alpha}} + \frac{\tilde{\alpha}}{m/\mu - \tilde{\alpha}}, \quad \mu \equiv \rho\pi r_1^2, \tag{5.9}$$

where $\alpha \equiv r_1 \tilde{\alpha} (\mathbf{z} \times \nabla\chi)$. If $\chi(x)$ is selected in the form

$$\chi'(r) = \frac{r_2 - r}{r_2 - r_1}, \tag{5.10}$$

then from (5.8) and (5.9) taking account of (5.7) it follows that

$$\tilde{\alpha} \leqslant \varkappa \equiv 1 - \frac{r_1}{r_2}; \tag{5.11}$$

$$-\frac{\gamma + 1}{\gamma - 1} \leqslant \frac{B}{Ar_1^2} \leqslant -1. \tag{5.12}$$

It follows from (4.11), (5.11), and (5.12) that by selecting the parameter

$$\tilde{\alpha} < \min\{\varkappa, m/\mu\},$$

we obtain condition (5.12) at the velocity profile (5.7) which is a sufficient condition for stability. It should be noted that sufficient conditions for stability (5.6) and (5.12) do not coincide. Inequality (5.12) is separated for a narrow class of stable flows of the form (5.7), but in the absence of (5.6) it is correct for the case when body mass is greater than the mass of liquid displaced by it, i.e., when condition (5.6) is upset.

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